

FINITE ELEMENT EXTERIOR CALCULUS FOR PARABOLIC PROBLEMS

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ABSTRACT. In this paper, we consider the extension of the finite element exterior calculus from elliptic problems, in which the Hodge Laplacian is an appropriate model problem, to parabolic problems, for which we take the Hodge heat equation as our model problem. The numerical method we study is a Galerkin method based on a mixed variational formulation and using as subspaces the same spaces of finite element differential forms which are used for elliptic problems. We analyze both the semidiscrete and a fully-discrete numerical scheme.

1. INTRODUCTION

In this paper we consider the numerical solution of the *Hodge heat equation*, the parabolic equation associated to the Hodge Laplacian. The initial-boundary value problem we study is

$$(1.1) \quad u_t + (d\delta + \delta d)u = f \quad \text{in } \Omega \times (0, T],$$

$$(1.2) \quad \text{tr}(\star u) = 0, \quad \text{tr}(\star du) = 0 \quad \text{on } \partial\Omega \times (0, T],$$

$$(1.3) \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega.$$

(We could consider other boundary conditions as well.) Here the domain $\Omega \subset \mathbb{R}^n$ has a piecewise smooth, Lipschitz boundary, the unknown u is a time dependent differential k -form on Ω , u_t denotes its partial derivative with respect to time, and d , δ , \star , and tr denote the exterior derivative, coderivative, Hodge star, and trace operators, respectively.

The numerical methods we consider are mixed finite element methods. These are based on the mixed weak formulation: find $(\sigma, u) : [0, T] \rightarrow H\Lambda^{k-1} \times H\Lambda^k$, such that $u(0) = u_0$ and

$$(1.4) \quad \langle \sigma, \tau \rangle - \langle d\tau, u \rangle = 0, \quad \tau \in H\Lambda^{k-1}, \quad t \in (0, T],$$

$$(1.5) \quad \langle u_t, v \rangle + \langle d\sigma, v \rangle + \langle du, dv \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k, \quad t \in (0, T],$$

(The notations are explained in the following section.) Notice that, unlike in the elliptic case, the harmonic forms need not be explicitly accounted for in the weak formulation. The well-posedness of the mixed formulation (1.4), (1.5) is established in a precise sense in Theorem 4.4.

In the simplest case of 0-forms ($k = 0$), the differential equation (1.1) is simply the heat equation, $u_t - \Delta u = f$, and the boundary condition (1.2) is the Neumann boundary

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condition, $\partial u / \partial n = 0$. Moreover, in this case the space $H\Lambda^{k-1}$ vanishes, and the weak formulation (1.4)–(1.5) is the usual (unmixed) one: $u : [0, T] \rightarrow H^1(\Omega)$ satisfies

$$\langle u_t, v \rangle + \langle \text{grad } u, \text{grad } v \rangle = \langle f, v \rangle, \quad v \in H^1(\Omega), \quad t \in (0, T].$$

In this case, the numerical methods and convergence results obtained in this paper reduce to ones long known [6, 10]. In the case of n -forms, the differential equation is again the heat equation, although the natural boundary condition is now the Dirichlet condition $u = 0$. In the case of n -forms, the weak formulation seeks $\sigma \in H(\text{div})$, $u \in L^2$ such that

$$\langle \sigma, \tau \rangle - \langle \text{div } \tau, u \rangle = 0, \quad \tau \in H(\text{div}), \quad \langle u_t, v \rangle - \langle \text{div } \sigma, v \rangle = \langle f, v \rangle, \quad v \in L^2, \quad t \in (0, T].$$

This mixed method for the heat equation was studied by Johnson and Thomée in [9] in two dimensions. Recently, Holst and Gillette [8] have studied this mixed method in n -dimensions using a finite element exterior calculus framework (in their work they consider hyperbolic problems as well).

For $k = 1$ or 2 in $n = 3$ dimensions, the differential equation (1.1) is the vectorial heat equation,

$$u_t + \text{curl curl } u - \text{grad div } u = f.$$

The weak formulations (1.4)–(1.5) for $k = 1$ and 2 correspond to two different mixed formulations of this equation, the former using the scalar field $\sigma = \text{div } u$ as the second unknown, the latter using the vector field $\sigma = \text{curl } u$. For $k = 1$, the boundary conditions (1.2), which are natural in the mixed formulation, become $u \cdot n = 0$, $\text{curl } u \times n = 0$, while for $k = 2$ these natural boundary conditions are $u \times n = 0$, $\text{div } u = 0$. This vectorial heat equation arises, for example, in the linearization of the Ginzburg–Landau equations for superconductivity [7], and is related to the dynamical equations of Stokes and Navier–Stokes flow (see, e.g., [11]).

To discretize (1.4), (1.5), we utilize the two main families of finite element differential forms, the $\mathcal{P}_r \Lambda^k$ and $\mathcal{P}_r^- \Lambda^k$ spaces. Between them they include lots of the best known families of finite elements on simplicial meshes [2, Section 5]. We give both semidiscrete and fully discrete schemes, and the corresponding convergence analysis. Convergence rates under different norms are shown in our final results (see Theorem 5.3 and Theorem 6.3 below). These achieve the optimal rates allowed by the finite element spaces provided some regularity assumptions are satisfied. These results also reveal the relation between convergence rates under different norms and the regularity of the exact solution.

The outline of the remainder of the paper is as follows. In Section 2, we review of basic notations from finite element exterior calculus, including the two main families of finite element differential forms, the $\mathcal{P}_r^- \Lambda^k$ and $\mathcal{P}_r \Lambda^k$ families, and some of their properties. In Section 3, we apply the elliptic theory to define an elliptic projection which will be crucial to the error analysis of the time-dependent problem, and to obtain error estimates for it. In Section 4, we turn to the Hodge heat equation at the continuous level and establish well-posedness of the mixed formulation. We then give a convergence analysis for the semidiscrete and fully discrete schemes in Sections 5 and 6, respectively. Finally, we present some numerical examples confirming the results.

2. PRELIMINARIES

We briefly review here some basic notions of finite element exterior calculus for the Hodge Laplacian. Details can be found in [1, § 2] and [2, §§ 3–4] and in numerous references given there.

For Ω a domain in \mathbb{R}^n and k an integer, let $L^2\Lambda^k = L^2\Lambda^k(\Omega)$ denote the Hilbert space of differential k -forms on Ω with coefficients in L^2 . This is the space of L^2 functions on Ω with values in $\text{Alt}^k \mathbb{R}^n$, a finite dimensional Hilbert space of dimension $\binom{n}{k}$ (understood to be 0 if $k < 0$ or $k > n$). The *Hodge star* operator \star is an isometry of $\text{Alt}^k \mathbb{R}^n$ and $\text{Alt}^{n-k} \mathbb{R}^n$, and so induces an isometry of $L^2\Lambda^k$ onto $L^2\Lambda^{n-k}$. The inner product in $L^2\Lambda^k$ may be written $\langle u, v \rangle = \int_{\Omega} u \wedge \star v$, with the corresponding norm denoted $\|u\|$. We view the exterior derivative $d = d^k$ as a unbounded operator from $L^2\Lambda^k$ to $L^2\Lambda^{k+1}$. Its domain, which we denote $H\Lambda^k(\Omega)$, consists of forms $u \in L^2\Lambda^k$ for which the distributional exterior derivative du belongs to $L^2\Lambda^{k+1}$. Assuming, as we shall, that Ω has Lipschitz boundary, the *trace operator* $\text{tr} = \text{tr}_{\partial\Omega}$ maps $H\Lambda^k(\Omega)$ boundedly into an appropriate Sobolev space on $\partial\Omega$ (namely $H^{-1/2}\Lambda^k(\partial\Omega)$). The *coderivative* δ is defined as $(-1)^{n(k+1)+1} \star d \star : H^*\Lambda^k \rightarrow H^*\Lambda^{k-1}$, where $H^*\Lambda^k := \star H\Lambda^{n-k}$. The adjoint $d^* = d_k^*$ of d^{k-1} is the unbounded operator $L^2\Lambda^k \rightarrow L^2\Lambda^{k-1}$ given by restricting δ the domain of d^* ,

$$D(d^*) = \mathring{H}^*\Lambda^k := \{u \in H^*\Lambda^k \mid \text{tr} \star u = 0\}.$$

We denote by \mathfrak{Z}^k and \mathfrak{Z}_k^* the null spaces of d^k and d_k^* , respectively. Their orthogonal complements in $L^2\Lambda^k$ are \mathfrak{B}_k^* and \mathfrak{B}^k , the ranges of d_{k+1}^* and d^{k-1} , respectively. The orthogonal complement of \mathfrak{B}^k inside \mathfrak{Z}^k is the space of *harmonic forms*

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^* = \{\omega \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \mid d\omega = 0, d^*\omega = 0\}.$$

The dimension of \mathfrak{H}^k is equal to the k th Betti number of Ω , so $\mathfrak{H}^k = 0$ for $k \neq 0$ if Ω is contractible. The *Hodge decomposition* of $L^2\Lambda^k$ and of $H\Lambda^k$ follow immediately:

$$(2.1) \quad L^2\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*,$$

$$(2.2) \quad H\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp},$$

where $\mathfrak{Z}^{k\perp} = H\Lambda^k \cap \mathfrak{B}_k^*$ denotes the orthogonal complement of \mathfrak{Z}^k in $H\Lambda^k$.

The *Hodge Laplacian* is the unbounded operator $L = dd^* + d^*d : D(L) \subset L^2\Lambda^k \rightarrow L^2\Lambda^k$ with

$$D(L) = \{v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k \mid d^*v \in H\Lambda^{k-1}, dv \in \mathring{H}^*\Lambda^{k+1}\}.$$

The null space of L consists precisely of the harmonic forms \mathfrak{H}^k .

For any $f \in L^2\Lambda^k$, there exists a unique solution $u = Kf \in D(L)$ satisfying

$$Lu = f \pmod{\mathfrak{H}}, \quad u \perp \mathfrak{H}^k,$$

(see [2, Theorem 3.1]). The solution u satisfies the Hodge Laplacian boundary value problem

$$(d\delta + \delta d)u = f - P_{\mathfrak{H}}f \text{ in } \Omega, \quad \text{tr} \star u = 0, \text{tr} \star du = 0 \text{ on } \partial\Omega,$$

together with side condition $u \perp \mathfrak{H}^k$ required for uniqueness. The solution operator K is a compact operator $L^2\Lambda^k \rightarrow H\Lambda^k \cap \mathring{H}^*\Lambda^k$ and *a fortiori*, is compact as an operator from $L^2\Lambda^k$ to itself.

Now we consider the mixed finite element discretization of the Hodge Laplacian boundary value problem, following [2]. This is based on the mixed weak formulation, which seeks $\sigma \in H\Lambda^{k-1}$, $u \in H\Lambda^k$, and $p \in \mathfrak{H}^k$ such that

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0, \quad \tau \in H\Lambda^{k-1}, \\ \langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle &= \langle f, v \rangle, \quad v \in H\Lambda^k, \\ \langle u, q \rangle &= 0, \quad q \in \mathfrak{H}^k. \end{aligned}$$

It admits a unique solution given by $u = Kf$, $\sigma = d^*u$, $p = P_{\mathfrak{H}}f$. We discretize the mixed formulation using Galerkin's method. For this, let Λ_h^{k-1} and Λ_h^k be finite dimension subspaces of $H\Lambda^{k-1}$ and $H\Lambda^k$, respectively, satisfying $d\Lambda_h^{k-1} \subset \Lambda_h^k$. We define the space of *discrete harmonic forms* \mathfrak{H}_h^k as the orthogonal complement of $\mathfrak{B}_h^k := d\Lambda_h^{k-1}$ inside $\mathfrak{Z}_h^k := \mathfrak{Z} \cap \Lambda_h^k$. This immediately gives the *discrete Hodge decomposition*

$$\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp},$$

where $\mathfrak{Z}_h^{k\perp}$ is the orthogonal complement of \mathfrak{Z}_h^k inside Λ_h^k .

The Galerkin method seeks $\sigma_h \in \Lambda_h^{k-1}$, $u_h \in \Lambda_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$\begin{aligned} \langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle &= 0, \quad \tau \in \Lambda_h^{k-1}, \\ \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p_h, v \rangle &= \langle f, v \rangle, \quad v \in \Lambda_h^k, \\ \langle u_h, q \rangle &= 0, \quad q \in \mathfrak{H}_h^k. \end{aligned} \tag{2.3}$$

For the analysis of this discretization, we require the existence of a third space $\Lambda_h^{k+1} \subset H\Lambda^{k+1}$ which contains $d\Lambda_h^k$, so that $\Lambda_h^{k-1} \xrightarrow{d} \Lambda_h^k \xrightarrow{d} \Lambda_h^{k+1}$ is a subcomplex of the segment $H\Lambda^{k-1} \xrightarrow{d} H\Lambda^k \xrightarrow{d} H\Lambda^{k+1}$ of the de Rham complex. Further we require that there exists a *bounded cochain projection*, i.e., bounded linear projection maps $\pi_h^j : H\Lambda^j \rightarrow \Lambda_h^j$, $j = k-1, k, k+1$, such that the diagram

$$\begin{array}{ccccc} H\Lambda^{k-1} & \xrightarrow{d} & H\Lambda^k & \xrightarrow{d} & H\Lambda^{k+1} \\ \pi_h^{k-1} \downarrow & & \pi_h^k \downarrow & & \pi_h^{k+1} \downarrow \\ \Lambda_h^{k-1} & \xrightarrow{d} & \Lambda_h^k & \xrightarrow{d} & \Lambda_h^{k+1} \end{array} \tag{2.4}$$

commutes. A key result of the finite element exterior calculus is that, under these assumptions, the Galerkin equations (2.3) admit a unique solution and provide a stable discretization.

Another important aspect of the finite element exterior calculus is the construction of finite element spaces Λ_h^k which satisfy these hypothesis, i.e., which combine to form de Rham subcomplexes with bounded cochain projections. Let there be given a shape regular family of meshes \mathcal{T}_h with mesh size h tending to 0. For each $r \geq 1$, we define two finite element subspaces of $H\Lambda^k$, denoted $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$. For $k = 0$, these two spaces coincide and equal the degree r Lagrange finite element subspace of $H^1(\Omega)$. For $k = n$, $\mathcal{P}_r^-\Lambda^n(\mathcal{T}_h)$ coincides with $\mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h)$, which may be viewed as the space of all piecewise polynomials of degree at most $r-1$, without inter-element continuity constraints. However, for $0 < k < n$,

$$\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r\Lambda^k(\mathcal{T}_h).$$

For stable mixed finite elements for the Hodge Laplacian, we have four possibilities (which reduce to just one for $k = 0$ and to two for $k = 1$ or n):

$$(2.5) \quad \Lambda_h^{k-1} = \left\{ \begin{array}{c} \mathcal{P}_r \Lambda^{k-1}(\mathcal{T}_h) \\ \text{or} \\ \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h) \end{array} \right\}, \quad \Lambda_h^k = \left\{ \begin{array}{c} \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h) \\ \text{or} \\ \mathcal{P}_{r-1} \Lambda^k(\mathcal{T}_h) \text{ (if } r > 1) \end{array} \right\}.$$

As the auxiliary space, if $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$, we take $\Lambda_h^{k+1} = \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{T}_h)$, while if $\Lambda_h^k = \mathcal{P}_{r-1} \Lambda^k(\mathcal{T}_h)$, we take $\Lambda_h^{k+1} = \mathcal{P}_{r-1}^- \Lambda^{k+1}(\mathcal{T}_h)$.

For this choice of spaces, it is known ([1, § 5.4], [2, § 5.5], [5]) that there exist cochain projections as in (2.4) for which $\pi_h^j : L^2 \Lambda^j \rightarrow \Lambda_h^j$ is bounded in $L^2 \Lambda^j$ uniformly with respect to h . In particular, this implies that there is a constant C independent of h such that

$$(2.6) \quad \|u - \pi_h^j u\| \leq C \inf_{v \in \Lambda_h^j} \|u - v\|, \quad u \in L^2 \Lambda^j.$$

Moreover, we have the approximation estimates

$$(2.7) \quad \|u - \pi_h^j u\| \leq Ch^s \|u\|_s, \quad 0 \leq s \leq \begin{cases} r, & \Lambda_h^j = \mathcal{P}_r^- \Lambda^j(\mathcal{T}_h), \\ r+1, & \Lambda_h^j = \mathcal{P}_r \Lambda^j(\mathcal{T}_h). \end{cases}$$

Note that we use $\|u\|_s$ as a notation for the Sobolev norm $\|u\|_{H^s \Lambda^j}$.

3. ELLIPTIC PROJECTION OF THE EXACT SOLUTION

As usual, we shall obtain error estimates for the finite element approximation to the evolution equation by comparing it to an appropriate elliptic projection of the exact solution into the finite element space. In this section we define the elliptic projection and establish error estimates for it.

Given any $u \in D(L)$, the elliptic projection of u is defined as $(\hat{\sigma}_h, \hat{u}_h, \hat{p}_h) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$, such that

$$(3.1) \quad \langle \hat{\sigma}_h, \tau \rangle - \langle d\tau, \hat{u}_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1},$$

$$(3.2) \quad \langle d\hat{\sigma}_h, v \rangle + \langle d\hat{u}_h, dv \rangle + \langle \hat{p}_h, v \rangle = \langle Lu, v \rangle, \quad v \in \Lambda_h^k,$$

$$(3.3) \quad \langle \hat{u}_h, q \rangle = \langle u, q \rangle, \quad q \in \mathfrak{H}_h^k.$$

By Theorem 3.8 of [2] there exists a unique solution to (3.1)–(3.3). Now we follow the approach of [2] to derive error estimates. To this end, we introduce some notations. First, let $P_{\mathfrak{H}_h} : L^2 \Lambda^k \rightarrow \mathfrak{H}_h^k$ denote the L^2 -projection. From (3.3), $P_{\mathfrak{H}_h} \hat{u}_h = P_{\mathfrak{H}_h} u$. Moreover, from [2, Section 3.4],

$$\hat{p}_h = P_{\mathfrak{H}_h}(Lu) = P_{\mathfrak{H}_h}(d\sigma),$$

where $\sigma = d^*u$, the last equality holding because $d^*du \in \mathfrak{B}_k^* \perp \mathfrak{Z}^k$, but $\mathfrak{H}_h^k \subset \mathfrak{Z}_h^k \subset \mathfrak{Z}^k$.

Next, define $\beta = \beta_h^k$, $\mu = \mu_h^k$, and $\eta = \eta_h^k$ by

$$\begin{aligned} \beta &= \|(I - \pi_h)K\|_{\mathcal{L}(L^2 \Lambda^k, L^2 \Lambda^k)}, \quad \mu = \|(I - \pi_h)P_{\mathfrak{H}_h^k}\|_{\mathcal{L}(L^2 \Lambda^k, L^2 \Lambda^k)}, \\ \eta &= \max_{j=0,1} \max [\|(I - \pi_h)dK\|_{\mathcal{L}(L^2 \Lambda^{k-j}, L^2 \Lambda^{k-j+1})}, \|(I - \pi_h)d^*K\|_{\mathcal{L}(L^2 \Lambda^{k+j}, L^2 \Lambda^{k+j-1})}]. \end{aligned}$$

From (2.6) and the compactness of $K : L^2\Lambda^k \rightarrow H\Lambda^k \cap \mathring{H}^*\Lambda^k$, we conclude that $\eta, \beta, \mu \rightarrow 0$ as $h \rightarrow 0$. Assuming H^2 regularity for the Hodge Laplacian (by which we mean both that $\|Kf\|_2 \leq C\|f\|_0$ for all $f \in L^2\Lambda^k$ and that $\mathfrak{H}^k \subset H^2$), then we have

$$(3.4) \quad \eta = O(h), \quad \beta, \mu = O(h^{\min(2,r)})$$

for any of the choices of spaces in (2.5). Note that $\beta = O(h^2)$ except in the case $r = 1$ and so $\Lambda_h^k = \mathcal{P}_1^-\Lambda^k$.

Finally, we denote the best approximation error in the L^2 norm by

$$E(w) = \inf_{v \in \Lambda_h^k} \|w - v\|, \quad w \in L^2\Lambda^k, \quad k = 0, \dots, n.$$

We are now ready to give the error estimates for the elliptic projection.

Theorem 3.1. *Let $u \in D(L)$ and let $(\hat{\sigma}_h, \hat{u}_h)$ be defined by (3.1)–(3.3). Then we have*

$$(3.5) \quad \|d(\sigma - \hat{\sigma}_h)\| \leq CE(d\sigma),$$

$$(3.6) \quad \|\sigma - \hat{\sigma}_h\| \leq C(E(\sigma) + \eta E(d\sigma)),$$

$$(3.7) \quad \|\hat{p}_h\| \leq C\mu E(d\sigma),$$

$$(3.8) \quad \|d(u - \hat{u}_h)\| \leq C(E(du) + \eta E(d\sigma)),$$

$$(3.9) \quad \|u - \hat{u}_h\| \leq C(E(u) + E(P_{\mathfrak{H}}u) + \eta[E(du) + E(\sigma)] + (\eta^2 + \beta)E(d\sigma) + \mu E(P_{\mathfrak{B}}u)).$$

Proof. This is essentially proven in [2], except that there it is assumed that $u \perp \mathfrak{H}$ and $\hat{u}_h \perp \mathfrak{H}_h$. To account for this difference, let $\tilde{u} = u - P_{\mathfrak{H}}u$ and $\tilde{u}_h = \hat{u}_h - P_{\mathfrak{H}_h}\hat{u}_h$. Then (3.1) and (3.2) continue to hold with u and u_h replaced by \tilde{u} and \tilde{u}_h , respectively, and, in place of (3.3), we have

$$\langle \tilde{u}_h, q \rangle = 0, \quad q \in \mathfrak{H}_h^k.$$

Application of Theorem 3.11 of [2] (with $f = Lu$ and $p = 0$) then gives the (3.5–3.8), and, instead of (3.9), we get

$$\begin{aligned} \|\tilde{u} - \tilde{u}_h\| &\leq C(E(\tilde{u}) + \eta[E(du) + E(\sigma)] + (\eta^2 + \beta)E(d\sigma) + \mu E(P_{\mathfrak{B}}u)) \\ &\leq C(E(u) + E(P_{\mathfrak{H}}u) + \eta[E(du) + E(\sigma)] + (\eta^2 + \beta)E(d\sigma) + \mu E(P_{\mathfrak{B}}u)). \end{aligned}$$

Thus $\|\tilde{u} - \tilde{u}_h\|$ is bounded by the right-hand side of (3.9), and, to complete the proof, it suffices bound $P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}\hat{u}_h$ by same quantity. Now

$$P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}\hat{u}_h = P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}u = (I - P_{\mathfrak{H}_h})P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u),$$

For the first term on the right-hand side, we use [2, Theorem 3.5] and (2.6) to get

$$\|(I - P_{\mathfrak{H}_h})P_{\mathfrak{H}}u\| \leq \|(I - \pi_h)P_{\mathfrak{H}}u\| \leq CE(P_{\mathfrak{H}}u).$$

To estimate the second term, we use the Hodge decomposition (2.2) to write $u - P_{\mathfrak{H}}u = u_b + u_{\perp}$ with $u_b \in \mathfrak{B}^k, u_{\perp} \in \mathfrak{Z}^{k\perp}$. Since $\mathfrak{H}_h^k \subset \mathfrak{Z}^k$, $P_{\mathfrak{H}_h}u_{\perp} = 0$, and since $\pi_h u_b \in \mathfrak{B}_h^k$, $P_{\mathfrak{H}_h}\pi_h u_b = 0$. Hence $P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u) = P_{\mathfrak{H}_h}(I - \pi_h)u_b$. We normalize this quantity by setting

$$q = P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u) / \|P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u)\| \in \mathfrak{H}_h^k.$$

Then $P_{\mathfrak{H}}q \in \mathfrak{H}$, and, by [2, Theorem 3.5], $\|q - P_{\mathfrak{H}}q\| \leq \|(I - \pi_h)P_{\mathfrak{H}}q\| \leq \mu$. Therefore,

$$\|P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u)\| = (P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u), q) = (P_{\mathfrak{H}_h}(I - \pi_h)u_b, q) = ((I - \pi_h)u_b, q).$$

Now $(I - \pi_h)u_b \in \mathfrak{B}^k$, and so is orthogonal to \mathfrak{H} . Thus

$$((I - \pi_h)u_b, q) = ((I - \pi_h)u_b, q - P_{\mathfrak{H}}q) \leq \|(I - \pi_h)u_b\| \|q - P_{\mathfrak{H}}q\| \leq C\mu E(P_{\mathfrak{B}}u),$$

by (2.6). Combining these results, we get

$$\|P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}\hat{u}_h\| \leq C[E(P_{\mathfrak{H}}u) + \mu E(P_{\mathfrak{B}}u)],$$

completing the proof of the theorem. \square

Assuming sufficient regularity of u and $\sigma = d^*u$, we can combine the estimates of the theorem with the approximation results of (2.7) to obtain rates of convergence for the elliptic projection. The precise powers of h and Sobolev norms that arise depend on the particular choice of spaces in (2.5). For example, if we take $\Lambda_h^{k-1} = \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h)$, then we can show the optimal estimate $\|\sigma - \sigma_h\| \leq Ch^{r+1}\|\sigma\|_{r+1}$, but, if $\Lambda_h^{k-1} = \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h)$, then clearly we can only have $\|\sigma - \sigma_h\| = O(h^r)$. Rather than give a complicated statement of the results, covering all the possible cases, in the following theorem and below we restrict to a particular choice of spaces from among the possibilities in (2.5). Moreover, we assume $r > 1$, since the case $r = 1$ is slightly different. However, very similar results can be obtained for any of the choices of spaces permitted in (2.5), including for $r = 1$, in the same way. Finally, we introduce the space

$$\bar{H}^r = \{u \in H^r \mid P_{\mathfrak{H}}u \in H^r, P_{\mathfrak{B}}u \in H^{r-2}\},$$

with the associated norm

$$\|u\|_{\bar{H}^r} = \|u\|_r + \|P_{\mathfrak{H}}u\|_r + \|P_{\mathfrak{B}}u\|_{r-2},$$

since it will arise frequently below.

Theorem 3.2. *Assume H^2 regularity for the Hodge Laplacian, so (3.4) holds and suppose that we use the finite element spaces $\Lambda_h^{k-1} = \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h)$ and $\Lambda_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$ (so that the auxilliary space is $\Lambda_h^{k+1} = \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}_h)$), for some $r > 1$. Then we have the following convergence rates for the elliptic projection:*

$$\begin{aligned} \|d(\sigma - \hat{\sigma}_h)\| &\leq Ch^r \|d\sigma\|_r, \\ \|\sigma - \hat{\sigma}_h\| &\leq Ch^r \|\sigma\|_r, \\ \|\hat{p}_h\| &\leq Ch^r \|d\sigma\|_{r-2}, \\ \|d(u - \hat{u}_h)\| &\leq Ch^r (\|du\|_r + \|d\sigma\|_{r-1}), \\ \|u - \hat{u}_h\| &\leq Ch^r \|u\|_{\bar{H}^r}. \end{aligned}$$

4. WELL-POSEDNESS OF THE PARABOLIC PROBLEM

We now turn to the Hodge heat equation. In this section we demonstrate well-posedness of the initial-boundary value problem (1.4), (1.5). The key tool is the Hille–Yosida–Phillips theory as presented, for example, in [3] and [4].

We begin by showing that the Hodge Laplacian is maximal monotone, or, equivalently, in the terminology of [4], that its negative is m-dissipative). This is the key hypothesis needed to apply the Hille–Yosida–Phillips theory to the problem (1.1)–(1.3).

Theorem 4.1. *The Hodge Laplacian L is maximal monotone. That is, it satisfies*

$$\langle Lv, v \rangle \geq 0, \quad \forall v \in D(L),$$

and, for any $f \in L^2\Lambda^k$, there exists $u \in D(L)$ such that $u + Lu = f$.

Proof. For any $v \in D(L)$, $\langle Lv, v \rangle = \langle dv, dv \rangle + \langle d^*v, d^*v \rangle$, so the monotonicity inequality is obvious. Now, for any $f \in L^2\Lambda^k$, the Riesz representation theorem furnishes a unique $u \in H\Lambda^k \cap \dot{H}^*\Lambda^k$ such that

$$(4.1) \quad \langle du, dv \rangle + \langle d^*u, d^*v \rangle + \langle u, v \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k \cap \dot{H}^*\Lambda^k.$$

We shall show that this u belongs to $D(L)$, from which it follows immediately that $u + Lu = f$.

To show that $u \in D(L)$, we must show that $du \in \dot{H}^*\Lambda^{k+1}$ and $d^*u \in H\Lambda^{k-1}$. From (4.1), $f - u$ is orthogonal to \mathfrak{H}^k , so, using the Hodge decomposition of $L^2\Lambda^k$, we may write $f - u = df_1 + d^*f_2$ with $f_1 \in H\Lambda^{k-1} \cap \mathfrak{B}_{k-1}^*$ and $f_2 \in \dot{H}^*\Lambda^{k+1} \cap \mathfrak{B}^{k+1}$. Then

$$\langle f - u, v \rangle = \langle df_1 + d^*f_2, v \rangle = \langle f_1, d^*v \rangle + \langle f_2, dv \rangle, \quad v \in H\Lambda^k \cap \dot{H}^*\Lambda^k.$$

Combining with (4.1), we get

$$(4.2) \quad \langle du - f_2, dv \rangle + \langle d^*u - f_1, d^*v \rangle = 0, \quad v \in H\Lambda^k \cap \dot{H}^*\Lambda^k.$$

Now $du, f_2 \in \mathfrak{B}^{k+1}$, so there exists $v \in \mathfrak{Z}^{k+1} = H\Lambda^k \cap \mathfrak{B}_k^*$ such that $dv = du - f_2$. Choosing this v in (4.2), we find $du = f_2 \in \dot{H}^*\Lambda^{k+1}$, as desired. Similarly $d^*u = f_1 \in H\Lambda^{k-1}$. \square

Since L is maximal monotone and self-adjoint, we obtain the following existence theorem. This is proved in [4] in Theorems 3.1.1 and 3.2.1 for $f = 0$ and $u_0 \in L^2\Lambda^k$, and in Proposition 4.1.6 for general f and u_0 in $D(L)$. Combining the two results by superposition, gives the theorem.

Theorem 4.2. *Suppose that $u_0 \in L^2\Lambda^k$ and $f \in C([0, T]; L^2\Lambda^k)$ are given and that either $f \in L^1((0, T); D(L))$ or $f \in W^{1,1}((0, T); L^2\Lambda^k)$. Then there exists a unique $u \in C([0, T]; L^2\Lambda^k) \cap C((0, T]; D(L)) \cap C^1((0, T]; L^2\Lambda^k)$, such that*

$$u_t + Lu = f \text{ on } \Omega \times (0, T], \quad u(0) = u_0.$$

If further, $u_0 \in D(L)$, then $u \in C([0, T]; D(L)) \cap C^1([0, T]; L^2\Lambda^k)$.

We denote by $S(t) : L^2\Lambda^k \rightarrow L^2\Lambda^k$ the solution operator for the homogeneous problem ($f \equiv 0$), so $u(t) = S(t)u_0$ solves $u_t + Lu = 0$, $u(0) = u_0$. Then $S(t)$ is a contraction in $L^2\Lambda^k$ for all $t \in [0, T]$, i.e., $\|S(t)\| \leq 1$, and $S(t)$ commutes with L on $D(L)$ (Theorem 3.1.1 of [4]).

We can measure the regularity of the solution (for general f) by using the iterated domains defined by $D(L^l) = \{u \in D(L^{l-1}) \mid L^{l-1}u \in D(L)\}$, $l \geq 2$. The next theorem show that if f is more regular, then the solution is also more regular.

Theorem 4.3. *Suppose that in addition to the hypotheses of Theorem 4.2, we have that f belongs to $C((0, T]; D(L)) \cap L^1((0, T); D(L^2))$. Then*

$$(4.3) \quad u \in C^1((0, T]; D(L)).$$

Proof. If $f = 0$, then [3, Theorem 7.7] implies that

$$u \in C^k((0, T]; D(L^l)),$$

for all $k, l \geq 0$. Therefore, it is sufficient to treat the case $u_0 = 0$, which we do using Duhamel's principle. By Proposition 4.1.6 of [4], the solution is given by

$$u(t) = \int_0^t S(t-s)f(s)ds$$

in this case, and, assuming that f satisfies the hypotheses of Theorem 4.2,

$$u \in C([0, T]; D(L)) \cap C^1([0, T]; L^2\Lambda^k).$$

Now $f \in L^1((0, T); D(L^2))$, so

$$L^2u(t) = \int_0^t S(t-s)L^2f(s)ds,$$

by the commutativity of $S(t-s)$ and L . Since $S(t-s)$ is a contraction in $L^2\Lambda^k$, this implies that $u \in C([0, T]; D(L^2))$ and so $Lu \in C([0, T]; D(L))$. Since we also assume that $f \in C((0, T]; D(L))$, (4.3) follows immediately from the equation $u_t = f - Lu$. \square

Next we show that the solution u guaranteed by Theorem 4.2, together with $\sigma = d^*u$, is a solution of the mixed problem (1.4), (1.5). Since $u \in C((0, T]; D(L))$, $\sigma = d^*u \in C((0, T]; H\Lambda^{k-1})$ and (1.4) holds. Clearly

$$\langle u_t, v \rangle + \langle Lu, v \rangle = \langle f, v \rangle, \quad v \in L^2\Lambda^k, \quad t \in (0, T].$$

Since $u \in C((0, T]; D(L))$, we have

$$\langle Lu, v \rangle = \langle dd^*u, v \rangle + \langle d^*du, v \rangle = \langle d\sigma, v \rangle + \langle du, dv \rangle, \quad v \in H\Lambda^k, \quad t \in (0, T].$$

Combining the last two equations gives (1.5).

We are now ready to state the main result for this section.

Theorem 4.4. *Suppose that $u_0 \in L^2\Lambda^k$ and $f \in C([0, T]; L^2\Lambda^k)$ are given and that either $f \in L^1((0, T); D(L))$ or $f \in W^{1,1}((0, T); L^2\Lambda^k)$. Then there exist unique*

$$\sigma \in C((0, T]; H\Lambda^{k-1}), \quad u \in C([0, T]; L^2\Lambda^k) \cap C((0, T]; D(L)) \cap C^1((0, T]; L^2\Lambda^k),$$

satisfying the mixed problem (1.4), (1.5) and the initial condition $u(0) = u_0$. If, moreover, the hypotheses of Theorem 4.3 are satisfied, then (4.3) holds.

Proof. We have already established existence. For uniqueness, we assume $f = 0$ and take $\tau = \sigma$ in (1.4) and $v = u$ in (1.5), to obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 = -\|\sigma\|^2 - \|du\|^2 \leq 0.$$

Therefore $\|u\|^2$ is decreasing in time, so if $u(0) = 0$, then $u \equiv 0$. Finally, (1.4) then implies that $\sigma \equiv 0$. \square

5. THE SEMIDISCRETE FINITE ELEMENT METHOD

The semidiscrete finite element method for the Hodge heat equation is Galerkin's method applied to the mixed variational formulation (1.4), (1.5). That is, we choose finite element spaces Λ_h^{k-1} and Λ_h^k as in (2.5) for some value of $r \geq 1$, and seek $(\sigma_h, u_h) \in C([0, T]; \Lambda_h^{k-1}) \times C^1([0, T]; \Lambda_h^k)$, such that $u_h(0) = u_h^0$, a given initial value in Λ_h^k , and

$$(5.1) \quad \langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \quad t \in (0, T],$$

$$(5.2) \quad \langle u_{h,t}, v \rangle + \langle d\sigma_h, v \rangle + \langle du_h, dv \rangle = \langle f, v \rangle, \quad v \in \Lambda_h^k, \quad t \in (0, T].$$

In this section we shall establish convergence estimates for this scheme.

We may interpret the semidiscrete solution in terms of two operators, $d_h^* : \Lambda_h^k \rightarrow \Lambda_h^{k-1}$ and $L_h : \Lambda_h^k \rightarrow \Lambda_h^k$, which are discrete analogues of d^* and L , respectively. For $v \in \Lambda_h^k$, $d_h^*v \in \Lambda_h^{k-1}$ is defined by the equation

$$\langle d_h^*v, \tau \rangle = \langle v, d\tau \rangle, \quad \tau \in \Lambda_h^{k-1},$$

and the discrete Hodge Laplacian $L_h : \Lambda_h^k \rightarrow \Lambda_h^k$ is given by $L_h = d_h^*d + dd_h^*$. The following characterization is then a direct consequence of the definitions.

Lemma 5.1. *The pair $(\sigma_h, u_h) \in C([0, T]; \Lambda_h^{k-1}) \times C^1([0, T]; \Lambda_h^k)$ solves (5.1) and (5.2) if and only if $u_h(t) \in C^1([0, T]; \Lambda_h^k)$ solves*

$$(5.3) \quad u_{h,t} + L_h u_h = P_h f, \quad 0 \leq t \leq T,$$

where P_h is L^2 projection of f onto Λ_h^k , and $\sigma_h = d_h^*u_h$.

From the theory of ordinary differential equations, there exists a unique solution $u_h \in C^1([0, T]; \Lambda_h^k)$ solving the ODE (5.3) and taking a given initial value. Letting $\sigma_h = d_h^*u_h$, we obtain a unique solution to the semidiscrete finite element scheme (5.1), (5.2).

Remark. The formulation (5.3) is useful for theoretical purposes, but is typically not implemented directly, rather only implicitly via the mixed method. This is because the operator d_h^* is not local. Even if the finite element function v is supported in just a few elements, d_h^*v will generally have global support.

Next, we turn to the convergence analysis. In Proposition 5.2 we shall give error estimates for the difference between the semidiscrete finite element solution and the elliptic projection of the exact solution of the evolution equations. Combining these estimates with the estimates from Section 3 for the elliptic projection gives error estimates for the semidiscrete finite element method, which we present in Theorem 5.3.

Assume the conditions of Theorem 4.3 hold, so the exact solution

$$u \in C([0, T]; L^2\Lambda^k) \cap C^1((0, T]; D(L)).$$

For each $t > 0$, we can then define the elliptic projection of $u(t)$ and of $u_t(t)$; see (3.1)–(3.3). Writing $(\hat{\sigma}_h(t), \hat{u}_h(t), \hat{p}_h(t))$ for the former, it is easy to see that its time-derivative, $(\hat{\sigma}_{h,t}, \hat{u}_{h,t}, \hat{p}_{h,t})$, is the elliptic projection of u_t . From Theorems 3.1 and 3.2 we obtain error estimates, such as

$$(5.4) \quad \|u_t - \hat{u}_{h,t}\| \leq C(E(u_t) + E(P_{\mathfrak{S}}u_t) + \eta[E(du_t) + E(\sigma_t)] \\ + (\eta^2 + \beta)E(d\sigma_t) + \mu E(P_{\mathfrak{B}}u_t)) \leq Ch^r \|u_t\|_{\bar{H}^r},$$

with the last inequality holding for the choice of spaces made in Theorem 3.2 (and similar results holding for the other allowable choices of spaces). Now, from (3.1),

$$(5.5) \quad \langle \hat{\sigma}_h, \tau \rangle - \langle d\tau, \hat{u}_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \quad t \in (0, T],$$

and, substituting $Lu = -u_t + f$ into (3.2),

$$(5.6) \quad \langle \hat{u}_{h,t}, v \rangle + \langle d\hat{\sigma}_h, v \rangle + \langle d\hat{u}_h, dv \rangle = \langle \hat{u}_{h,t} - u_t, v \rangle + \langle f, v \rangle - \langle \hat{p}_h, v \rangle.$$

Define

$$\Sigma_h = \hat{\sigma}_h - \sigma_h, \quad U_h = \hat{u}_h - u_h,$$

the difference between the elliptic projection and the finite element solution. Subtracting (5.1) and (5.2) from (5.5) and (5.6), respectively, gives

$$(5.7) \quad \langle \Sigma_h, \tau \rangle - \langle d\tau, U_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \quad 0 < t \leq T,$$

$$(5.8) \quad \langle U_{h,t}, v \rangle + \langle d\Sigma_h, v \rangle + \langle dU_h, dv \rangle = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, v \rangle, \quad v \in \Lambda_h^k, \quad 0 < t \leq T.$$

We shall now use these equations to derive bounds on Σ_h and U_h in terms of $\hat{u}_{h,t} - u_t$ and \hat{p}_h , for which we derived bounds in Section 3. In the remainder of the paper, we adopt the notation $\|\cdot\|_{L^\infty(L^2)}$ for the norm in $L^\infty(0, T; L^2\Lambda^k(\Omega))$ and similarly for other norms.

Proposition 5.2. *Assume $u_0 \in D(L)$. Then*

$$\begin{aligned} \|U_h\|_{L^\infty(L^2)} + \|\Sigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} &\leq C(\|U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}), \\ \|\Sigma_h\|_{L^\infty(L^2)} + \|d\Sigma_h\|_{L^2(L^2)} &\leq C(\|d_h^* U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^2(L^2)}). \end{aligned}$$

Proof. By Theorem 4.2, $u \in C([0, T]; D(L)) \cap C^1([0, T]; L^2\Lambda^k)$. For each $t \in (0, T]$, take $\tau = \Sigma_h(t) \in \Lambda_h^{k-1}$ in (5.7) and $v = U_h(t) \in \Lambda_h^k$ in (5.8), and add to obtain

$$(5.9) \quad \frac{1}{2} \frac{d}{dt} \|U_h\|^2 + \|\Sigma_h\|^2 + \|dU_h\|^2 = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, U_h \rangle,$$

which implies

$$\frac{d}{dt} \|U_h\|^2 \leq 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\| \|U_h\|.$$

Taking $t^* \in [0, T]$ such $\|U_h\|_{L^\infty(L^2)} = \|U_h(t^*)\|$, and integrating this inequality from 0 to t^* gives

$$\|U_h(t^*)\|^2 \leq \|U_h(0)\|^2 + 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)} \|U_h\|_{L^\infty(L^2)},$$

whence

$$(5.10) \quad \|U_h\|_{L^\infty(L^2)} \leq \|U_h(0)\| + 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)},$$

which gives the desired bound on U_h . To get the bound on Σ_h and dU_h , integrate (5.9) over $t \in [0, T]$. This gives

$$\|\Sigma_h\|_{L^2(L^2)}^2 + \|dU_h\|_{L^2(L^2)}^2 \leq \frac{1}{2} \|U_h(0)\|^2 + \|U_h\|_{L^\infty(L^2)} \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)},$$

and so, by (5.10),

$$\|\Sigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} \leq C(\|U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}),$$

which completes the proof of the first inequality.

To prove the second inequality, we differentiate (5.7) in time and take $\tau = \Sigma_h \in \Lambda_h^{k-1}$, and then add to (5.8) with $v = d\Sigma_h \in \Lambda_h^k$ (here we use the subcomplex property $d\Lambda_h^{k-1} \subset \Lambda_h^k$). This gives

$$\frac{1}{2} \frac{d}{dt} \|\Sigma_h\|^2 + \|d\Sigma_h\|^2 = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, d\Sigma_h \rangle.$$

By integrating in time, first over $[0, t^*]$ with $t^* \in [0, T]$ chosen so that $\|\Sigma_h\|_{L^\infty(L^2)} = \|\Sigma_h(t^*)\|$, and then over all of $[0, T]$, we deduce that

$$\|\Sigma_h\|_{L^\infty(L^2)} + \|d\Sigma_h\|_{L^2(L^2)} \leq C(\|\Sigma_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^2(L^2)}).$$

Finally, we note from (5.1) and (3.1) that $\Sigma_h = d_h^* U_h$, and so complete the proof. \square

Now suppose, for simplicity, that we choose the initial data u_h^0 to equal the elliptic projection of u_0 . Then $U_h(0) = 0$ and the right-hand sides of the inequalities in Proposition 5.2 simplify. Bounding them using Theorem 3.2 and (5.4) we get, for the choice of spaces indicated in the theorem,

$$\begin{aligned} \|U_h\|_{L^\infty(L^2)} + \|\Sigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} &\leq Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|dd^*u\|_{L^1(H^{r-2})}), \\ \|\Sigma_h\|_{L^\infty(L^2)} + \|d\Sigma_h\|_{L^2(L^2)} &\leq Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|dd^*u\|_{L^2(H^{r-2})}). \end{aligned}$$

Combining these estimates with the estimates in Theorem 3.2 for the elliptic projection, we obtain the main result of the section.

Theorem 5.3. *Suppose that, in addition to the hypotheses of Theorem 3.2 and 4.3, $u_0 \in D(L)$. Let (σ, u) be the solution of (1.4), (1.5) satisfying (1.3), and (σ_h, u_h) the solution of (5.1), (5.2) with the spaces selected as in Theorem 3.2 and $u_h(0)$ chosen to be equal to the elliptic projection of u_0 . Then, we have the following error estimates for the semidiscrete finite element method:*

$$\begin{aligned} \|\sigma - \sigma_h\|_{L^2(L^2)} &\leq Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|d^*u\|_{L^2(H^r)}), \\ \|\sigma - \sigma_h\|_{L^\infty(L^2)} &\leq Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}), \\ \|d(\sigma - \sigma_h)\|_{L^2(L^2)} &\leq Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|dd^*u\|_{L^2(H^r)}), \\ \|u - u_h\|_{L^\infty(L^2)} &\leq Ch^r(\|u\|_{L^\infty(\bar{H}^r)} + \|u_t\|_{L^1(\bar{H}^r)}), \\ \|d(u - u_h)\|_{L^2(L^2)} &\leq Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|du\|_{L^2(H^r)} + \|dd^*u\|_{L^2(H^{r-1})}). \end{aligned}$$

6. THE FULLY DISCRETE FINITE ELEMENT METHOD

If we combine the semidiscrete finite element method with a standard time-stepping scheme to solve the resulting system of ordinary differential equations, we obtain a fully discrete finite element method for the Hodge heat equation (1.4), (1.5). For simplicity, we use backward Euler's method with constant time step $\Delta t = T/M$. We may choose any of the pairs of finite element spaces indicated in (2.5) for any value of $r \geq 1$, but, as above, for simplicity we restrict ourselves to the choice $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$ and $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ with $r > 1$, the results for the other cases being simple variants. The fully discrete method seeks $\sigma_h^n \in \Lambda_h^{k-1}$, $u_h^n \in \Lambda_h^k$, satisfying the equations

$$(6.1) \quad \langle \sigma_h^n, \tau \rangle - \langle d\tau, u_h^n \rangle = 0, \quad \tau \in \Lambda_h^{k-1},$$

$$(6.2) \quad \left\langle \frac{u_h^n - u_h^{n-1}}{\Delta t}, v \right\rangle + \langle d\sigma_h^n, v \rangle + \langle du_h^n, dv \rangle = \langle f(t^n), v \rangle, \quad v \in \Lambda_h^k.$$

for $1 \leq n \leq M$. It is easy to see that this linear system for u_h^n, σ_h^n is invertible at each time step. We initialize by choosing $u_h^0 \in \Lambda_h^k$. We also define $\sigma_h^0 \in \Lambda^{k-1}$ so that (6.1) holds for $n = 0$.

Next, we turn to the convergence analysis. We first obtain error estimates for the difference between the fully discrete finite element solution and the elliptic projection of the exact solution of the evolution equations. These are stated in (6.10) and (6.11). Combining these estimates with the estimates from Section 3 for the elliptic projection, we obtain the error estimates for the fully discrete finite element method presented in Theorem 6.3.

The analysis is similar to that for the semidiscrete finite element method, but with some extra complications arising from the time discretization. Let $(\hat{\sigma}_h^n, \hat{u}_h^n, \hat{p}_h^n)$ be the elliptic projection of $u^n = u(t^n)$.

Now, from (3.1)

$$(6.3) \quad \langle \hat{\sigma}_h^n, \tau \rangle - \langle d\tau, \hat{u}_h^n \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \quad 0 \leq n \leq M,$$

and, from (3.2) and the equation $u_t + Lu = f$,

$$(6.4) \quad \begin{aligned} \langle \frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\Delta t}, v \rangle + \langle d\hat{\sigma}_h^n, v \rangle + \langle d\hat{u}_h^n, dv \rangle &= \langle \frac{\hat{u}_h^n - \hat{u}_h^{n-1}}{\Delta t} - u_t^n, v \rangle + \langle f^n, v \rangle - \langle \hat{p}_h^n, v \rangle \\ &= \langle \frac{(\hat{u}_h^n - u^n) - (\hat{u}_h^{n-1} - u^{n-1})}{\Delta t}, v \rangle + \langle \frac{u^n - u^{n-1}}{\Delta t} - u_t^n, v \rangle + \langle f^n, v \rangle - \langle \hat{p}_h^n, v \rangle, \end{aligned}$$

$v \in \Lambda_h^k, \quad 1 \leq n \leq M.$

Set

$$\Sigma_h^n = \hat{\sigma}_h^n - \sigma_h^n, \quad U_h^n = \hat{u}_h^n - u_h^n,$$

the difference between the elliptic projection and the finite element solution at each time step. Subtracting (6.1) and (6.2) from (6.3) and (6.4), respectively, gives

$$(6.5) \quad \langle \Sigma_h^n, \tau \rangle - \langle d\tau, U_h^n \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \quad 0 \leq n \leq M,$$

and

$$(6.6) \quad \begin{aligned} \langle \frac{U_h^n - U_h^{n-1}}{\Delta t}, v \rangle + \langle d\Sigma_h^n, v \rangle + \langle dU_h^n, dv \rangle \\ = \langle \frac{(\hat{u}_h^n - u^n) - (\hat{u}_h^{n-1} - u^{n-1})}{\Delta t}, v \rangle + \langle \frac{u^n - u^{n-1}}{\Delta t} - u_t^n, v \rangle - \langle \hat{p}_h^n, v \rangle \\ = \langle z^n, v \rangle, \quad v \in \Lambda_h^k, \quad 1 \leq n \leq M, \end{aligned}$$

where $z^n \in L^2\Lambda^k$ is defined by the last equation. We easily see that

$$\|z^n\| \leq \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \|(\hat{u}_{h,t} - u_t)(s)\| ds + \frac{\Delta t}{2} \|u_{tt}\|_{L^\infty(L^2)} + \|\hat{p}_h^n\|.$$

By Theorem 3.2, the last term on the right hand side is bounded by $Ch^r \|dd^*u\|_{L^\infty(H^{r-2})}$, and, by (5.4), the first term on the right hand side by

$$\frac{Ch^r}{\Delta t} \|u_t\|_{L^1([t^{n-1}, t^n], \bar{H}^r)}.$$

Thus we have proved:

Proposition 6.1.

$$\|z^n\| \leq \frac{\Delta t}{2} \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|dd^*u\|_{L^\infty(H^{r-2})} + \frac{1}{\Delta t} \|u_t\|_{L^1([t^{n-1}, t^n], \bar{H}^r)}).$$

We shall now use equations (6.5) and (6.6) to derive bounds on Σ_h and U_h in terms of z . Toward this end we adopt the notations

$$\|f\|_{l^\infty(X)} = \max_{1 \leq n \leq M} \|f^n\|_X, \quad \|f\|_{l^2(X)} = (\Delta t \sum_{n=1}^M \|f^n\|_X^2)^{1/2}, \quad \|f\|_{l^1(X)} = \Delta t \sum_{n=1}^M \|f^n\|_X.$$

Proposition 6.2. *Assume $u_0 \in D(L)$. Then*

$$\begin{aligned} \|U_h\|_{l^\infty(L^2)} + \|\Sigma_h\|_{l^2(L^2)} + \|dU_h\|_{l^2(L^2)} &\leq C(\|U_h(0)\| + \|z\|_{l^1(L^2)}), \\ \|\Sigma_h\|_{l^\infty(L^2)} + \|d\Sigma_h\|_{l^2(L^2)} &\leq C(\|d_h^*U_h(0)\| + \|z\|_{l^2(L^2)}). \end{aligned}$$

Proof. Take $\tau = \Sigma_h^n \in \Lambda_h^{k-1}$ in (6.5), $v = U_h^n \in \Lambda_h^k$ in (6.6), and add to obtain

$$(6.7) \quad \Delta t (\|\Sigma_h^n\|^2 + \|dU_h^n\|^2) + \|U_h^n\|^2 = (U_h^{n-1} + \Delta t z^n, U_h^n),$$

which implies

$$\|U_h^n\| \leq \|U_h^{n-1}\| + \Delta t \|z^n\|.$$

By iteration,

$$(6.8) \quad \|U_h\|_{l^\infty(L^2)} \leq \|U_h^0\| + \Delta t \sum_{n=1}^M \|z^n\|,$$

which is the desired bound on U_h . To get the bound on Σ_h and dU_h , we derive from (6.7) that

$$\frac{1}{2} \|U_h^n\|^2 - \frac{1}{2} \|U_h^{n-1}\|^2 + \Delta t (\|\Sigma_h^n\|^2 + \|dU_h^n\|^2) \leq (\Delta t z^n, U_h^n) \leq \Delta t \|z^n\| \|U_h\|_{l^\infty(L^2)}.$$

Summing then gives

$$\frac{1}{2} \|U_h^M\|^2 - \frac{1}{2} \|U_h^0\|^2 + \|\Sigma_h\|_{l^2(L^2)}^2 + \|dU_h\|_{l^2(L^2)}^2 \leq \|U_h\|_{l^\infty(L^2)} \|z\|_{l^1(L^2)},$$

and so, by (6.8),

$$\|\Sigma_h\|_{l^2(L^2)}^2 + \|dU_h\|_{l^2(L^2)}^2 \leq \|U_h^0\|^2 + \frac{3}{2} \|z\|_{l^1(L^2)}^2,$$

which completes the proof of the first inequality.

To prove the second inequality, we take $\tau = \Sigma_h^n \in \Lambda_h^{k-1}$ in (6.5) at both time level $n-1$ and level n . This gives

$$(6.9) \quad (\Sigma_h^{n-1}, \Sigma_h^n) = (d\Sigma_h^n, U_h^{n-1}), \quad (\Sigma_h^n, \Sigma_h^n) = (d\Sigma_h^n, U_h^n), \quad 1 \leq n \leq M.$$

Next take $v = d\Sigma_h^n \in \Lambda_h^k$ in (6.6) and substitute (6.9) to get

$$(\Sigma_h^n - \Sigma_h^{n-1}, \Sigma_h^n) + \Delta t \|d\Sigma_h^n\|^2 = \Delta t (z^n, d\Sigma_h^n), \quad 1 \leq n \leq M,$$

whence

$$\|\Sigma_h^n\|^2 - \|\Sigma_h^{n-1}\|^2 + \Delta t \|d\Sigma_h^n\|^2 \leq \Delta t \|z^n\|^2.$$

Again we get a telescoping sum, so

$$\|\Sigma_h\|_{l^\infty(L^2)}^2 + \|d\Sigma_h\|_{l^2(L^2)}^2 \leq C(\|\Sigma_h^0\|^2 + \|z\|_{l^2(L^2)}^2).$$

This implies the second inequality and so completes the proof of the proposition. \square

As in Section 5, we choose the initial data u_h^0 to equal the elliptic projection of u_0 for simplicity. Then $U_h(0) = 0$ and the right-hand sides of the inequalities in Proposition 6.2 simplify. Bounding them via Proposition 6.1 we get for the first

$$(6.10) \quad \|U_h\|_{l^\infty(L^2)} + \|\Sigma_h\|_{l^2(L^2)} + \|dU_h\|_{l^2(L^2)} \\ \leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|dd^*u\|_{L^\infty(H^{r-2})} + \|u_t\|_{L^1(\bar{H}^r)}).$$

For the second, we bound the $L^1([t^{n-1}, t^n])$ norm in Proposition 6.1 by Δt times the L^∞ norm, and substitute the resulting bound for z in the second estimate of Proposition 6.2, obtaining

$$(6.11) \quad \|\Sigma_h\|_{l^\infty(L^2)} + \|d\Sigma_h\|_{l^2(L^2)} \leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|dd^*u\|_{L^\infty(H^{r-2})} + \|u_t\|_{L^\infty(\bar{H}^r)}).$$

Combining (6.10), (6.11) with the estimates in Theorem 3.2 for the elliptic projection, we obtain the main result of the section.

Theorem 6.3. *Under the same assumptions as Theorem 5.3, Let (σ, u) be the solution of (1.4), (1.5) satisfying (1.3), and (σ_h^n, u_h^n) the solution of (6.1), (6.2) with u_h^0 equal to the elliptic projection of u_0 . Then, we have the following error estimates for the fully discrete finite element method:*

$$\begin{aligned} \|\sigma - \sigma_h\|_{l^2(L^2)} &\leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|u_t\|_{L^1(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}), \\ \|\sigma - \sigma_h\|_{l^\infty(L^2)} &\leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|u_t\|_{L^\infty(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}), \\ \|d(\sigma - \sigma_h)\|_{l^2(L^2)} &\leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|u_t\|_{L^\infty(\bar{H}^r)} + \|dd^*u\|_{L^\infty(H^r)}), \\ \|u - u_h\|_{l^\infty(L^2)} &\leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|u\|_{L^\infty(\bar{H}^r)} + \|u_t\|_{L^1(\bar{H}^r)}), \\ \|d(u - u_h)\|_{l^2(L^2)} &\leq C\Delta t \|u_{tt}\|_{L^\infty(L^2)} + Ch^r (\|u_t\|_{L^1(\bar{H}^r)} + \|du\|_{L^\infty(H^r)} + \|dd^*u\|_{L^\infty(H^{r-1})}). \end{aligned}$$

The error estimates are analogous to those of Theorem 5.3 for the semidiscrete solution, with each containing an additional $O(\Delta t)$ term coming from the time discretization. For each quantity, the error is of order $O(\Delta t + h^r)$.

7. NUMERICAL EXAMPLES

In this section, we present the results of simple numerical computations verifying the theory above.

First we compute a two-dimensional example for the 1-form Hodge heat equation. Using vector proxies, we may write the parabolic equations (1.1)–(1.3) as

$$\begin{aligned} u_t + (\text{curl rot} - \nabla \text{div})u &= f \text{ in } \Omega \times [0, T], \\ u \cdot n = \text{rot } u &= 0 \text{ on } \partial\Omega \times [0, T], \quad u(\cdot, 0) = u_0 \text{ in } \Omega, \end{aligned}$$

where

$$\text{rot } u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \text{curl } u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1} \right).$$

We choose Ω to be a square annulus $[0, 1] \times [0, 1] \setminus [0.25, 0.75] \times [0.25, 0.75]$ and take the exact solution as

$$u = \begin{pmatrix} 100x(x-1)(x-0.25)(x-0.75)t \\ 100y(y-1)(y-0.25)(y-0.75)t \end{pmatrix}.$$

Note that this function is not orthogonal to 1-harmonic forms on Ω . We use the finite element spaces $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ (Lagrange elements of degree r) for $\sigma = -\operatorname{div} u$ and $\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$ (Raviart–Thomas elements) for u , starting with an initial unstructured mesh, and then refining it uniformly. We take $\Delta t = 0.0001$ and compute the error at time $T = 0.01$ (after 100 time steps). Tables 1 and 2 show the results for $r = 1$ and 2 respectively. The rates of convergence are just as predicted by the theory.

mesh size	$\ \sigma - \sigma_h\ $	rate	$\ \nabla(\sigma - \sigma_h)\ $	rate	$\ u - u_h\ $	rate
h	0.0008490	1.99	0.1026276	1.01	0.0010586	0.96
$h/2$	0.0002132	1.99	0.0512846	1.00	0.0005341	0.99
$h/4$	0.0000534	2.00	0.0256528	1.00	0.0002678	1.00
$h/8$	0.0000133	2.00	0.0128295	1.00	0.0001340	1.00

TABLE 1. Computation with $P_1\Lambda^0 \times P_1^-\Lambda^1$ in two dimensions.

mesh size	$\ \sigma - \sigma_h\ $	rate	$\ \nabla(\sigma - \sigma_h)\ $	rate	$\ u - u_h\ $	rate
h	0.0000093	3.03	0.0016510	2.03	0.0000705	1.97
$h/2$	0.0000012	3.00	0.0004119	2.00	0.0000178	1.99
$h/4$	0.0000001	3.00	0.0001031	2.00	0.0000045	1.99
$h/8$	0.0000000	3.04	0.0000258	2.00	0.0000011	2.00

TABLE 2. Computation with $P_2\Lambda^0 \times P_2^-\Lambda^1$ in two dimensions.

For the second example, we let Ω be the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ in \mathbb{R}^3 , and again solve the 1-form Hodge heat equation. Using vector proxies, the initial–boundary value problem becomes

$$\begin{aligned} u_t + (\operatorname{curl} \operatorname{curl} - \nabla \operatorname{div})u &= f \text{ in } \Omega \times [0, T] \\ u \cdot n &= 0, \operatorname{curl} u \times n = 0 \text{ on } \partial\Omega \times [0, T], \quad u(\cdot, 0) = u_0 \text{ in } \Omega. \end{aligned}$$

We take the exact solution to be

$$u = \begin{pmatrix} \sin(\pi x_1)t \\ \sin(\pi x_2)t \\ \sin(\pi x_3)t \end{pmatrix}.$$

Table 3 shows the errors and rates of convergence for linear elements on a sequence of uniform meshes, again at time $T = 0.01$ after 100 time steps. Once again, the rates of convergence are just as predicted by the theory.

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mesh size	$\ \sigma - \sigma_h\ $	rate	$\ \nabla(\sigma - \sigma_h)\ $	rate	$\ u - u_h\ $	rate
0.25	0.0023326	2.06	0.0260155	1.02	0.0026024	1.00
0.125	0.0005735	2.02	0.0134836	0.95	0.0013499	0.95
0.0625	0.0001429	2.01	0.0068169	0.98	0.0006879	0.97

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